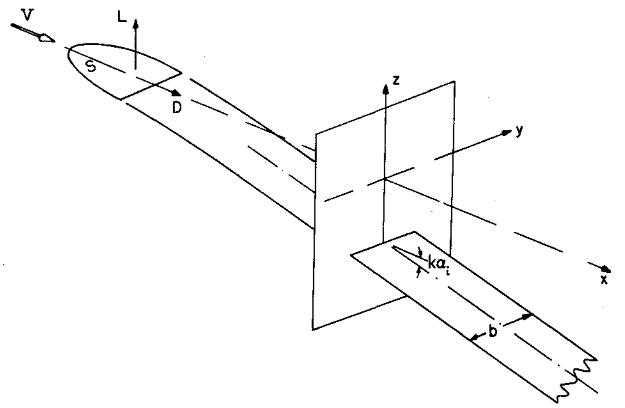

Fig. 1. Sketch showing wing, trailing-vortex wake, and "Trefftz Plane."

presumed to be unknown; viz., we shall write for the downwash (uniform across the span) far downstream

$$\text{Downwash far downstream} = k\frac{C_L}{\pi R} = 2k\frac{L}{\pi\rho V b^2}$$

$$= k\alpha_i, \text{ say} \quad (1)$$

The x-forces acting on the system are 1) the thrust, equal and opposite to the drag D, and 2) the resultant of the pressure perturbations acting on a transverse yz-plane downstream. The change of x-impulse effected by these forces is also calculated by integration over this transverse plane. The momentum balance therefore reads

$$-D - \iint(p - p_\infty)dydz = \rho V\iint(u - V)dydz \quad (2)$$

Now, the pressure is given by Bernoulli's Equation

$$p - p_\infty = \frac{\rho}{2}\{V^2 - (u^2 + v^2 + w^2)\} \quad (3)$$

where u, v, w are the usual Cartesian velocity components and the flow field far downstream consists of two-dimensional flow in planes normal to the trailing-vortex sheet, which is inclined downward at the angle $k\alpha_i$ mentioned above. Thus, u, v, and w can be expressed in terms of the velocity potential $\varphi(y, z + xk\alpha_i)$, say, of this two-dimensional flow. To second order, they are

$$u = V + \varphi_z k\alpha_i, \quad v = \varphi_y, \quad w = \varphi_z \quad (4)$$

so that Eq. (3) becomes, to second order

$$p - p_\infty = -\frac{\rho}{2}\{2V\varphi_z k\alpha_i + \varphi_y^2 + \varphi_z^2\} \quad (5)$$

and Eq. (2) becomes

$$-D + \rho V k\alpha_i\iint\varphi_z dydz + \frac{\rho}{2}\iint(\varphi_y^2 + \varphi_z^2)dydz$$

$$= \rho V k\alpha_i\iint\varphi_z dydz \quad (6)$$

or

$$D = \frac{\rho}{2}\iint(\varphi_y^2 + \varphi_z^2)dydz \quad (7)$$

But this is exactly the statement relating drag to kinetic energy, mentioned above; we have therefore arrived at the familiar result for elliptic lift distribution[1,2]

$$D = (C_L/\pi R)L = (2L^2/\pi\rho V^2 b^2) \quad (8)$$

The first interesting feature of this calculation is that the induced angle $k\alpha_i$ far downstream has disappeared from consideration. The x-impulse term on the right-hand side is $k\alpha_i$ times the z-impulse, which is $-L$; thus it is $-k$ times the drag; nevertheless it is cancelled by a pressure contribution in the left-hand side, and the correct answer is obtained for any value of k.

A second interesting feature is that the net pressure contribution is a force directed *upstream*. This follows from what has just been said, for the first integral on the left-hand side of Eq. (6) is $-kD$, namely $-2D$, and the second integral is $+D$. Thus one's usual concept—at least the author's—wherein the drag is balanced by reduced pressures downstream, seems to be incorrect. The correct picture involves generally *increased* pressures downstream, accompanied by upwind-directed induced velocities.

It is also instructive (and makes a fine homework problem!) to carry out this calculation in a stream-fixed frame of reference. It must then be remembered that the trailing-vortex wake is moving downward in this frame, and unsteady-flow formulas must be used to obtain the correct pressure. When this is done, the results are, of course, identical to those of the present note.

Finally, the generalization to a wing-system having arbitrary spanwise lift distribution can easily be carried out. As long as u is given by a second-order perturbation added to V, as is true far downstream for all small-perturbation cases, the x-impulse integral on the right-hand side of Eq. (2) is cancelled by a term in the pressure integral, and Eq. (7) results. This calculation is made by Landau and Lifshitz,[3] but their argument has serious flaws, such as the claim that the x-impulse integral

$$\rho\iint(u - V)dydz$$

is zero. As we have seen, it is actually proportional to the lift L in the classical case of elliptic lift distribution. The error arises from insufficient care in applying the principles of continuity and momentum in an infinite domain.

References

[1]von Kârmân, T. and Burgers, J., *General Aerodynamic Theory—Perfect Fluids*. Vol. II *Aerodynamic Theory*, edited by W. F. Durand, Sec. IIIB, Springer-Verlag, Berlin, 1935, pp. 127 and 128.

[2]*Incompressible Aerodynamics*, edited by B. Thwaites, Oxford, New York, 1960, p. 304.

[3]Landau, L. D. and Lifshitz, E. M., *Fluid Mechanics*, Pergamon Press, New York, 1959, pp. 174, 176.

Errata

On the Dependence of Materials Erosion on Environmental Parameters at Supersonic Velocities

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THE following footnote was inadvertently omitted: "Received September 28, 1972; revision received August 9, 1973."

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